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A SURVEY ON RELATION BETWEEN THE RAMANUJAN'S CONGRUENCE FORM AND THE BACK BACKGROUND OF THE MODULUS FORM

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ABSTRACT

In this paper, we will be covering the background of modular forms and explain how they are used to prove many important results about partition function congruence. We first go over the background of molecular forms mainly focusing on modular forms over $SL_2(z)$. In addition to this, we will also introduce more general modular forms and half-integral weight molecular forms by Ramanujan's congruence proof. Apart from this, an additional family of Congruences was proven using the theory of modular forms over $SL_2(z)$ and also using some computer-checkable computation, we will also modify the Method of ono to give a more general framework of proof. It allows us to straightforwardly extend the congruence on proving modules 13.17, 19, and 23 to give similar congruence to modules 29 and 31.

Keywords: Partition function, Multi partition, Ramanujan Congruence, Modulo Prime Powers.

I. INTRODUCTION

In this paper, we will be introducing the theory of modular form. For this, we define a Modular form and explain some of its applications. A modular form is a holomorphic function on the upper half plane H that satisfies Certain functional equations and boundedness conditions as we approach $i\infty$ or as we approach Q from certain directions. The Simplest example of the modular form, That will be dealing with significantly, a modular form over SL_2 (Z) Which is a holomorphic function f: H → C Such that f(z) is uniformly bounded in the region Im (Z) >1 and

$$f \frac{pz+q}{mz+n} = f(z)\,(mz+n)^r$$

for all Z ∈ H and all p,q,m,n ∈ z such that pn - mq = 1 Apart from studying the partition function, modular forms have been immensely useful in other areas, they are used in determining the Congruence of other functions Such as Ramanujan's tau function. In other areas of number theory, modular forms have proven useful in determining the number of ways to write an integer n as the sum of k^2, format's last Theorem.

II. THE MODULAR ARMP AND CONGRUENCE SUBGROUPS

Let us consider for some ring R, $SL_2(R)$ be the group of 2X2 matrices with coefficients in R and determinant 1, and let GL_2 (R) be the group of invertible 2X2 matrices. Moreover define the upper half plane H to be H = { Z ∈ C : Im(Z) > 0}
It turns out that $GL_2(R)$ and consequently $SL_2(R)$ and $SL_2(R)$ have a natural group action on the upper half plane H.

If $\gamma = {}_{mn}^{pq} \in GL_2(R)$ then γ acts on the upper half plane according as Mobius's transformation

$$\gamma z = {}_{mn}^{pq} Z := \frac{pz+q}{mz+n} \in H$$

It is straightforward to verify that this indeed forms a group action. This fact will be crucial in defining modular forms. Here negative of the identity matrix – I performs a trivial action on H. Hence it is sometimes natural to consider the group actions of PSL_2 (R) = $SL_2(R/\pm L)$ and $PSL_2(Z) = SL_2(Z)/(\pm L)$. The latter is often called a modular group.
Now, Consider this important result about SL_2 (z).

Theorem 2.2

The group $SL_2(Z)$ is generated by the two matrices Sand T where

$$S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ ---------- (1)}$$

Modular forms are holomorphic functions on H that Satisfy a certain functional equation with respect to a chosen discrete subgroup of SL_2 (R) . For the purpose of this exposition, we have only the discrete subgroups $SL_2(Z)$ as well as certain types of congruence subgroups. there are finite index subgroups of $SL_2(Z)$, often represented by ϒ , such that there exists some positive integer N such that Ker(φ_n) ⊂ ϒ for φ_n : SL_2 (Z) → SL_2 (Z/ixj Z) is the obvious map of reducing each element of the matrix modulo N. The most important congruence subgroups are

$$\Upsilon(N) = \begin{pmatrix} p & q \\ m & n \end{pmatrix} \in SL_2(Z) : P_n \equiv 1 \text{ (mod N)}$$

$$q, m \equiv O \text{ (mod N)} = \ker(\varphi_n)$$

$$\Upsilon_,(N) = \begin{pmatrix} p & q \\ m & n \end{pmatrix} \in SL_2(Z) : pn \equiv 1 \text{ (mod N)} \quad C \equiv \text{ (mod N)}$$

$$\Upsilon_0(N) = \begin{pmatrix} p & q \\ m & n \end{pmatrix} \in SL_2(Z) : C \equiv \text{ (mod N)}$$

It is clear that

ϒ (N) ⊂ ϒ, (N) ⊂ Υ_0(N) ⊂ $SL_2(Z)$

The level of a congruence subgroup ϒ is dependent as the smallest N such that ϒ(N) ⊂ ϒ. Thus one can verify that Υ_I(N) and Υ_0 (N) are both of level N

Modular forms are $SL_2(Z)$

Let us define a modular form over $SL_2(Z)$

Definition 2:3 A holomorphic function f ! H → C is a modular form of weight K if for all

$$\Upsilon = \begin{pmatrix} p & q \\ m & n \end{pmatrix} \in SL_2(Z)$$

$$F(y_Z) := f \frac{pz+q}{mz+n} = (mz+n)^n f(Z) \text{ ------------(2)}$$

And if f(z) is uniformly bounded in the region lim (Z) > 1. Condition (2) is often called the modularity condition.

2.4 The partition function and congruences

The integer partition function ρ(n) equals the number of ways to write n as the sum of positive integers in nondecreasing order. with ρ(0)defined to be 1. For instance ρ(5)= 7 since 5 can be written as 5=5, 5 = 1+4 , 5 = 2+3, 5 = 1+2+2 , 5 = 1+1+3, 5 = 1+1+1+2 and 5 = 1+1+1+1+1 . Despite the simplicity of the partition function.

No closed-form expression for the partition function is known. However, there exists a known asymptotic formula

$$F(x) = \frac{1}{4n-3} \exp \pi \frac{\Upsilon}{\frac{2n}{3}} \{1 + 0(1)\}$$

As n → ∞ which was first proven by Hardy and Ramanujan.

Ramanujan also discovered that the partition function has surprising congruence patterns. For instance, he proved the following, now called the Ramanujan's congruence.

$$\rho(5n + 4) \equiv 0 \text{ (mod 5)} \quad \text{---------------- (3)}$$

$$\rho(7n + 5) \equiv 0 \text{ (mode 7)} \quad \text{--------------- (4)}$$

$$\rho(11n + 6) \equiv 0 \text{ (mode 11)} \quad \text{-------------- (5)}$$

Ramanujan postulated that there are no equally simple congruence properties, as the three listed above. this has now been proven by which means that if l is some prime 0 ≤ ∞ ≤ 1 , -1 is some integer and ρ (ln+β) ≡ 0 (mode1) is true for all non-negative integer n then we must have (1, β) equals either (5,4), (7,5) or (11,6) However this does not mean there are no other congruence of the partition function Result of Watson and Atkin give us that if 24m ≡1 (mod 5^p 7^q 11^m) for p,q,m ≥ 0 then $\rho(t) \equiv 0$ (mod 5^p $7^{\frac{q+2}{2}}$ 11^m) moreover there have been proven congruence modulus primes aapartfrom 5, 7,& 11 after results by authors including At kin and O' Brien which proved results such as ρ (11^3 : 13n + 237) ≡ 0 (mod 13)

And ρ(59^4 ; 13n +1112247) ≡ 0 (mod 13)

III. HECKE OPERATORS OVER $M_K(SL_2(\mathbb{Z}))$ AND ANOTHER FAMILY OF PARTITION FUNCTION CONGRUENCES

In this section, we will show another family of congruences relating to the partition function that can be established solely using the theory of modular forms in $\mathbf{M_k(SL_2(\mathbb{Z}))}$ While many more general congruences can be established using the more general theory of modular forms (such as half-integral weight forms or forms with Nebentypus characters), we will primarily deal with such congruences in Section 6. We mostly follow several of the results in [18] that only require the theory of modular forms in $\mathbf{M_k(SL_2(\mathbb{Z}))}$, and will see that we can still prove many fascinating congruences. These congruences will primarily deal with primes from 13 through 31, and all follow a very similar set of calculations. Specifically, we will get congruences which give us information about

$$p\left(\frac{\ell^k.(24n+r_{\ell,k})+1}{24}\right) \text{ (mod } \ell),$$

for primes 13 ≤ ℓ ≤ 31 and all positive integers k, where r ℓ,k is the unique integer between 0 and 23 such that $\ell^k \, r_{\ell,k} \equiv -1$ (mod 24). The method followed generally follows that presented in Section 4 of [18], though we significantly change the way this is presented to explain a more general theory behind the congruences we are proving. We also note that cases 13, 17, 19, and 23 are done in [18], and although cases = 29, and 31 were not done, they follow the same approach.

IV. THE PARTITION FUNCTION AND THE RAMANUJAN CONGRUENCES

Recall that the partition function p is an arithmetic function such that p(0) = 1 and p(n) for n ∈ N is the number of ways to represent n as the sum of at most n positive integers, where order does not matter. As a result, we can think of p(n) as the number of ways to write p(n) = a1·1+a2·2+a3·3+· · · for nonnegative integers a1, a2, a3, ... since all that matters is the number of times we have each integer in our sum. Therefore, we have the formal power series

$$\sum_{n=0}^{\infty} p(n)p^n = \prod_{i=1}^{\infty}(1 + q^i + q^{2i} + \ldots\ldots) = \prod_{i=1}^{\infty}(1 - q^i)^{-1}$$

The right-hand side can be seen to converge to a nonzero value for |q| < 1 by taking logarithms, so it is true for all $q = e^{2\pi iz}$, z ∈ ℍ.. We also have the following theorem by Jacobi.

Definition 4.1. Let η(z) be the function such that

$$\eta^{24}(z) = \Delta(z).$$

Specifically, we choose the root such that

$$\eta(z) = e^{i\pi z/12} \cdot \prod_{n=1}^{\infty}(1 - q^n) = e^{i\pi z/12} \cdot \left(\sum_{n=0}^{\infty} p(n)q^n\right)^{-1}$$

The above results give a strong reason to suggest the theory of modular forms can be quite useful in understanding the partition function. In fact, many of the results that have been established on partition function congruences involve establishing that certain functions are modular forms and relating these functions to the partition function. Our goal in this section is to prove three famous congruences proven by Ramanujan. Specifically, we will show that for all integers n ≥ 0, we have

p(5n + 4) ≡ 0 (mod 5)
p(7n + 5) ≡ 0 (mod 7)
p(11n + 6) ≡ 0 (mod 11).

There exist several known proofs of these congruences. We will show a proof of all three congruences which only requires the theory of modular forms over SL2(Z).

Proposition 4.2. We have that θ_k is a linear operator and satisfies the product rule for modular forms f ∈ M_k , g ∈ M_{k^I} . Proof. The fact that θk is linear is straightforward to verify. To show the product rule, note that fg ∈ M_{k+k^I} SL_2 (ℤ). Therefore,

$$\Theta_{k+k^I} f(g) = \frac{1}{2\pi i}\frac{d}{dz}(fg) - \frac{(k+k^I)}{12} E_2 (fg)$$

$$= \frac{1}{2\pi i}\left(f\frac{d}{dz}g + g\frac{d}{dz}f\right) - \left(\frac{k}{12}E_2 f\right)g - \left(\frac{k^I}{12}E_2 g\right)f$$

$$= (\theta_k f)g + (\theta_{k^I} g)f$$

as desired.

4.3 The Ramanujan's Congruences

Now we are going to establish Ramanujan's congruence definitions.

The general method of proving Ramanujan's congruences.

Our prime I ∈ {5,7,11}

We consider that $\emptyset^{\frac{t^2-1}{24}}$ (why ∅ for I = 5, $\emptyset^2$ for I = 7 , $\emptyset^5$ for I = 11) is congruent modulo I I to D(f) which is the polynomial of P, Q and R of some δ.

I am looking q series expansions of P. Q.R and using Ramanujan's derivatives Q, identities this Q means conferment of $\emptyset^{\frac{t^2-1}{24}}$ must be O module ℓ .

In 1919 paper Ramanujan's p roved two congruence using q – pochhammer symbol notation

$$\sum_{k=0}^{\infty} p(5k + 4)\, q^k = \frac{5\,(q^5)5_\infty}{(q^6)\infty}$$

$$\sum_{k=0}^{\infty} p(7k + 5)\, q^k = \frac{7\,(q^7)3_\infty}{(q^4)\infty} + 49q\frac{(q^7)7_\infty}{(q^8)\infty}$$

It appears there are no equally simple properties for any module involving primes other than there in 1920, GH Hardy focused on three unpublished manuscripts. Of Ramanujan's p(h) (Ramanujan 1921) the proof of this means scripts employ the Einstein Series.

4.4 Theorem For all n ∈ Z and all r ∈ Z

(i) $P_{2+5r}(5n+3) \equiv 0 \pmod 5$
(ii) $P_{1+5r}(5n+4) \equiv 0 \pmod 5$
(iii) $P_{1+7k}(7n+5) \equiv 0 \pmod 7$
(iv) $P_{1+7k}(7n+6) \equiv 0 \pmod 7$
(v) $P_{8+11r}(11n+4) \equiv 0 \pmod{11}$

(vi) $P_{1+11r}(11n+6) \equiv 0 \pmod{11}$

(vii) $P_{3+11r}(11n+7) \equiv 0 \pmod{11}$

(viii) $P_{6+11r}(11n+8) \equiv 0 \pmod{11}$

(ix) $P_{7+11r}(11n+9) \equiv 0 \pmod{11}$

(x) $P_{10+13r}(13n+8) \equiv 0 \pmod{13}$

(xi) $P_{11+5^2r}(5^2n+14) \equiv 0 \pmod{5^2}$

(xii) $P_{6+5^2r}(5^2n+19) \equiv 0 \pmod{5^2}$

(xiii) $P_{1+5^2r}(5^2n+24) \equiv 0 \pmod{5^2}$

(xiv) $P_{95+11^2r}(11^2n+9) \equiv 0 \pmod{11^2}$

(xv) $P_{64+11^2r}(11^2n+64) \equiv 0 \pmod{11^2}$

(xvi) $P_{7+11^2r}(11^2n+86) \equiv 0 \pmod{11^2}$

(xvii) $P_{29+11^2r}(11^2n+97) \equiv 0 \pmod{11^2}$

(xviii) $P_{51+11^2r}(11^2n+108) \equiv 0 \pmod{11^2}$

(xix) $P_{73+11^2r}(11^2n+119) \equiv 0 \pmod{11^2}$

V.CONCLUSION

Ramanujan's Modulus theory of partition is far earlier than other partition theories whose proofs are given in our assumption and explanation.

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