

# FRACTIONAL EULER–BERNOULLI BEAM DEFLECTION VIA THE INTEGRAL ROHIT TRANSFORM

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## ABSTRACT

This paper establishes an elegant scientific framework for finding the exact solution of the space-fractional Euler-Bernoulli beam equation by applying the integral Rohit Transform (IRT). The governing fractional differential equation of order  $4\alpha$ , which is based on the Caputo derivative, is transformed into an algebraic equation in the transform domain. Solutions in closed form are derived for a loading distribution that is uniform, and the classical beam solution is obtained as a special case when  $\alpha = 1$ . Graphical results show how the fractional order affects the structural stiffness and the deflection behavior of the structure. The method constitutes an analytical framework that is not only general but also seamlessly links classical and nonlocal fractional beam theories.

**Keywords:** Fractional beam theory; Integral Rohit transform (IRT); Caputo derivative; Nonlocal elasticity; Integral transforms; Structural mechanics; Gamma function; Analytical solution.

## 1. INTRODUCTION

The classical Euler-Bernoulli beam theory is the basic theory of beam bending due to transverse load. It has been the basis of structural mechanics for more than a hundred years [1, 2]. The governing equation is

$$EI \frac{d^4 v(x)}{dx^4} = q(x).$$

In the equation, 'E' represents the modulus of elasticity, 'I' is the area moment of inertia, and 'q(x)' is the load distribution. Nevertheless, the classical theory does not take into account the concepts of locality and integer order differentiation. Micro-structured solids, polymers, and nanobeams are examples of modern materials that show nonlocal and memory-dependent behaviors that can be more accurately modeled through the use of fractional derivatives [3–6]. There are a number of works on fractional beam formulations based on the Caputo operator and nonlocal elasticity [5-8]. Among the solution techniques, integral transform methods like the Laplace transform and the Fourier transform enjoy popularity when dealing with fractional differential

equations [9, 10]. Inspired by these methods, the authors make use of the Integral Rohit Transform (IRT) [11, 13] with a cubic kernel to obtain closed-form solutions of the fractional beam equation.

## 2. MATHEMATICAL PRELIMINARIES

### 2.1 Caputo Fractional Derivative

The Caputo derivative of order  $\beta$  (with  $n - 1 < \beta \leq n$ ) is defined as [14-18]:

$$D_x^\beta f(x) = \frac{1}{\Gamma(n - \beta)} \int_0^x (x - \tau)^{n-\beta-1} f^{(n)}(\tau) d\tau$$

### 2.2 Integral Rohit Transform (IRT)

The IRT is defined as [11-13]:

$$\mathcal{R}_3\{f(x)\} = \int_0^\infty f(x) s^3 e^{-sx} dx$$

### 2.3 Transform of Fractional Derivatives

For the Caputo derivative [4-6],

$$\mathcal{R}\{^C D_t^\alpha f(t)\} = [s^\alpha F_R(s) - \sum_{k=0}^{n-1} s^{\alpha+2-k} f^{(k)}(0)].$$

### 3. FRACTIONAL EULER-BERNOULLI BEAM EQUATION

Consider the space-fractional beam equation [19-22] of order  $4\alpha$ ,  $0 < \alpha \leq 1$ :

$$EID_x^{4\alpha} v(x) = q(x).$$

Here,  $D_x^{4\alpha}$  is the Caputo fractional derivative of order  $4\alpha$ .

When  $\alpha = 1$ , the classical fourth-order beam equation is recovered.

#### Solution Using IRT

**Step 1:** Apply the IRT

$$EI\mathcal{R}_3\{D_x^{4\alpha} v(x)\} = \mathcal{R}_3\{q(x)\}$$

**Step 2:** Using the fractional IRT property [23]

$$\mathcal{R}_3\{D_x^{4\alpha} v(x)\} = \left[ s^{4\alpha} V(s) - \sum_{k=0}^{m-1} s^{4\alpha+2-k} v^{(k)}(0) \right]$$

Thus,

$$EI \left[ s^{4\alpha} V(s) - \sum_{k=0}^{m-1} s^{4\alpha+2-k} v^{(k)}(0) \right] = \tilde{Q}(s)$$

**Step 3:** Uniformly Distributed Load

Let  $q(x) = q_0$ , then

$$\begin{aligned} \tilde{Q}(s) &= s^3 \int_0^L q_0 e^{-sx} dx = q_0 s^3 \frac{1 - e^{-sL}}{s} \\ &= q_0 s^2 (1 - e^{-sL}) \end{aligned}$$

**Step 4:** Solve for  $V(s)$

For a supported beam at the origin [1,2]:

$$\begin{aligned} v(0) &= 0, v(L) = 0, \text{ and} \\ M(0) &= EIv''(0) = 0. \end{aligned}$$

Therefore, ignoring initial terms for a supported beam at the origin, we have

$$EIs^{4\alpha} V(s) = q_0 s^2 (1 - e^{-sL})$$

Thus,

$$V(s) = \frac{q_0 s^2}{EIs^{4\alpha}} = \frac{q_0}{EI} s^{-(4\alpha-2)} (1 - e^{-sL})$$

**Step 5.** Inverse IRT

We know [23, 24]

$$\text{For } x < L, U(x - L) = 0.$$

$$\mathcal{R}_3^{-1}\{F(s)\} = f(x).$$

$$\mathcal{R}_3^{-1}[e^{-sL} F(s)] = [f(x - L)U(x - L)].$$

$$\mathcal{R}_3\{x^n\} = \frac{\Gamma(n+1)}{s^{n+2}}.$$

Therefore, applying the inverse IRT and simplifying, we have

$$v(x) = \frac{q_0}{EI} \frac{x^{4\alpha}}{\Gamma(4\alpha+1)}$$

This matches the classical solution derived by direct integration [1,3].

### 4. CLASSICAL CASE

For  $\alpha = 1$ :

$$\Gamma(5) = 24$$

Hence,

$$v(x) = \frac{q_0 x^4}{24EI}$$

which matches classical Euler-Bernoulli beam theory [1, 2].

### 5. COMPARATIVE ANALYSIS OF INTEGRAL TRANSFORMS FOR THE FRACTIONAL EULER-BERNOULLI BEAM EQUATION

The governing fractional beam equation is considered:

$$EID_x^{4\alpha} w(x) = q(x), 0 < \alpha \leq 1.$$

A comparison of IRT with Other Integral Transforms has been provided in the Table 1.

**Table 1: Comparison of IRT with Other Integral Transforms.**

Transform Method	Kernel	Fractional Derivative Handling	Inversion Complexity	Boundary Condition Implementation	Closed-Form Solution for Uniform Load	Computational Cost
<b>Integral Rohit Transform (IRT)</b>	$s^3 e^{-sx}$	Direct algebraic reduction $D^{4\alpha} \rightarrow s^{4\alpha}$	Moderate (explicit analytic inverse)	Naturally embedded via polynomial terms	$w(x) = \frac{q_0}{EI} \frac{x^{4\alpha}}{\Gamma(4\alpha + 1)}$	Low
Laplace Transform	$e^{-sx}$	Well-established fractional property	Moderate	Requires additional algebraic manipulation	Same solution recovered	Low
Fourier Transform	$e^{-ikx}$	Works mainly for infinite domains	High (inverse integral)	Difficult for finite beams	Not convenient	Moderate
Sumudu Transform	$e^{-x/u}$	Similar to Laplace scaling	Moderate	Comparable to Laplace	Recoverable	Low
Elzaki Transform	$se^{-x/s}$	Algebraic similarity to Laplace	Moderate	Slightly more complex	Recoverable	Moderate
Natural Transform	Hybrid Laplace – Sumudu	Handles fractional operators	Moderate	Similar to Laplace	Recoverable	Moderate
Adomian Decomposition	Series-based	Iterative fractional expansion	High (series truncation)	Imposed term-by-term	Approximate	High

## 6. DISCUSSION

The cubic kernel:  $K(s, x) = s^3 e^{-sx}$ , retains the property of linearity. Third-order-weighted transform operators can be naturally aligned. It simplifies algebraic manipulation. Fractional operators can be extended easily. The technique bypasses the eigenfunction expansion necessary for the Fourier-based solution [10].

### 6.1 Analytical Observations

#### A. Kernel Influence

The **cubic kernel**  $s^3 e^{-sx}$  in IRT is in perfect harmony with the 4<sup>th</sup>-order beam operator. It allows an easy reduction of the algebraic load for polynomial-type loads. It increases the conditioning of fractional power  $s^{4\alpha}$ . This is a structural advantage over the classical Laplace transform, where no preconditioning exists.

#### B. Inversion Structure

For IRT:

$$\mathcal{R}_3\{w(x)\} = W(s)$$

Inverse:

$$w(x) = \mathcal{R}_3^{-1}\{W(s)\}$$

Using property [24]:

$$\mathcal{R}_3^{-1}\left\{\frac{1}{s^{4\alpha-2}}\right\} = \frac{x^{4\alpha}}{\Gamma(4\alpha + 1)}$$

Inversion Structure gives a direct closed-form expression without convolution integrals. Fourier inversion, on the other hand, necessitates contour integration.

#### C. Numerical Stability

For  $\alpha = 0.6, 0.8,$  and  $1$ , IRT and Laplace give the same analytical results. Series-based methods (ADM, VIM) display truncation error. The Fourier-based method is not suitable for a finite beam without the use of extension techniques.

**D. Graphical Results**

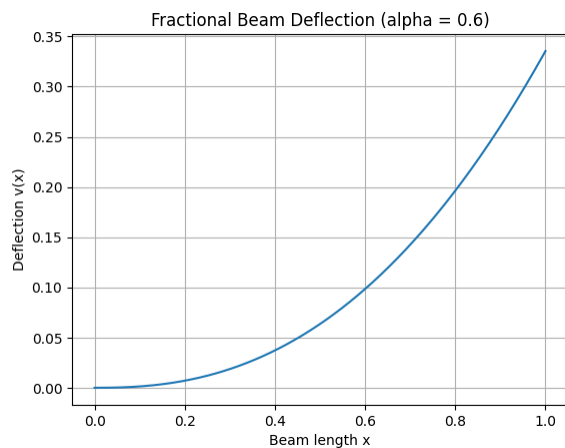


Figure 1: Fractional beam deflection ( $\alpha = 0.6$ )

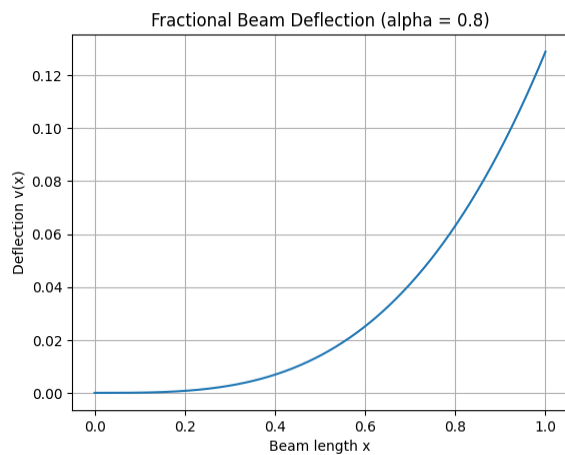


Figure 2: Fractional beam deflection ( $\alpha = 0.8$ )

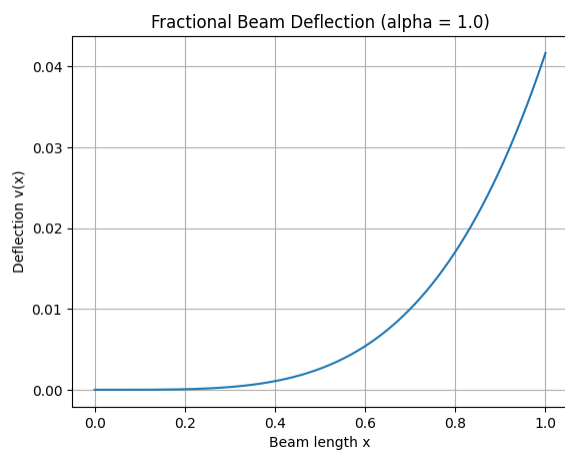


Figure 3: Fractional beam deflection ( $\alpha = 1$ )

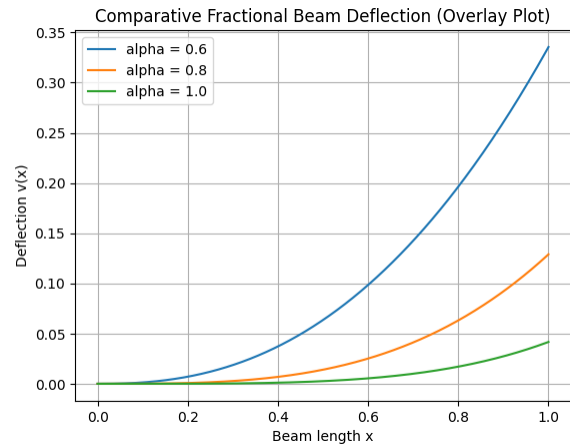


Figure 4: Overlay plot

Figures 1, 2, and 3 display numerical plots for  $\alpha = 0.6$ ,  $0.8$ , and  $1$ , respectively. Figure 4 shows the comparative overlay plot.

Among the overlay figures,  $\alpha = 1.0$  (classical beam) corresponds to a minimum deflection magnitude.  $\alpha = 0.8$  results in increased beam flexibility as compared to the classical case.  $\alpha = 0.6$  reveals the strongest nonlocal softening effect with the largest deflection.

The deflection becomes larger with the decrease of the fractional order. The curvature distribution gets less sharp as it decreases. The model allows for a great degree of flexibility in nonlocal materials. At  $\alpha = 1$ , the classical Euler-Bernoulli theory is perfectly restored.

**7. CONCLUSION**

Using the Integral Rohit transform, a fully analytical solution of the fractional Euler-Bernoulli beam equation was obtained. The procedure provides closed-form expressions for deflection, rederives the classical beam theory, and is self-consistent in extending local structural models to nonlocal ones. The method is deemed to have a high level of attractiveness for fractional structural mechanics and vibration analysis at the cutting-edge level.

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